

3.1A Critical Numbers are the numbers that make the 1<sup>st</sup> derivative equal 0 or undefined.

1.  $f(x) = x^2 + 2x - 4$

$$f'(x) = 2x + 2$$

$$0 = 2x + 2$$

$$\frac{-2}{2} = \frac{-2x}{2}$$

$$-1 = x$$

2.  $f(x) = x^2(x-3)$

$$f(x) = x^3 - 3x^2$$

$$f'(x) = 3x^2 - 6x$$

$$f'(x) = x(3x - 6)$$

$$0 = x(3x - 6)$$

$$x = 0$$

$$3x - 6 = 0$$

$$+6 \quad +6$$

$$3x = 6$$

$$x = 2$$

$$3. f(x) = \frac{x}{x-2}$$

$$f'(x) = \frac{1(x-2) - 1(x)}{(x-2)^2}$$

$$f'(x) = \frac{x-2-x}{(x-2)^2}$$

$$f'(x) = \frac{-2}{(x-2)^2}$$

$$(x-2)^2 \cdot 0 = \frac{-2}{(x-2)^2} \cdot (x-2)^2$$

$0 = -2$  no x's make  $f'(x) = 0$

$f'(x)$  is undefined at 2

$$4. f(x) = \frac{2x}{x^2+1}$$

$$f'(x) = \frac{2(x^2+1) - (2x)(2x)}{(x^2+1)^2}$$

$$f'(x) = \frac{2x^2+2-4x^2}{(x^2+1)^2}$$

$$f'(x) = \frac{-2x^2+2}{(x^2+1)^2}$$

$$(x^2+1)^2 \cdot 0 = \frac{-2x^2+2}{(x^2+1)^2} \cdot (x^2+1)^2$$

$$0 = \frac{-2x^2+2}{-2}$$

$$\frac{-2}{-2} = \frac{-2x^2}{-2}$$

$$1 = x^2 \quad \pm \sqrt{1} = x \quad x = \pm 1$$

3.1B Evaluate  $f(x)$  at the interval endpoints; the critical numbers  
the highest  $f(x)$  is your max, the lowest is your min.

5.  $f(x) = 2(3-x)$   $[-1, 2]$

find critical numbers:

$$f'(x) = -2$$

$0 = -2 \rightarrow$  no critical numbers

evaluate  $f(x)$ :

$x$	$-1$	$2$
$f(x)$	$8$	$2$

$(-1, 8)$ : max

$(2, 2)$ : min

6.  $f(x) = x^2 + 2x - 4$   $[-1, 1]$

$$f'(x) = 2x + 2$$

$$0 = 2x + 2$$

$-1 = x \leftarrow$  critical number also an end point

$x$	$-1$	$1$
$f(x)$	$-5$	$-1$

$(-1, -5)$ : min.

$(1, -1)$ : max.

$$7. f(x) = x^3 - \frac{3}{2}x^2 \quad [-1, 2]$$

$$f'(x) = 3x^2 - 3x$$

$$0 = 3x(x-1)$$

$$x = 0, 1$$

$x$	$-1$	$0$	$1$	$2$
$f(x)$	$-\frac{5}{2}$	$0$	$-\frac{1}{2}$	$2$

$(-1, -\frac{5}{2})$ : min.

$(2, 2)$ : max

$$8. f(x) = \frac{x}{x-1} \quad [2, 4]$$

$$f'(x) = \frac{1(x-1) - 1(x)}{(x-1)^2} = \frac{\cancel{x-1} - x}{(x-1)^2}$$

$$f'(x) = \frac{-1}{(x-1)^2}$$

undefined at  $x=1$ , (not in our interval)

$x$	2	4
$f(x)$	2	$\frac{4}{3}$

$(2, 2)$ : max.

$(4, \frac{4}{3})$ : min.

3.2 Plug in the endpoints of the interval  $[a, b]$   
if  $f(a) = f(b)$  then there is a number  
in the interval where  $f'(c) = 0$

9.  $f(x) = x^2 - 5x + 4$

$$f(1) = 1^2 - 5(1) + 4 = 0 \quad \checkmark$$

$$f(4) = 4^2 - 5(4) + 4 = 0$$

$$f'(x) = 2x - 5$$

$$0 = 2x - 5$$

$$\frac{5}{2} = x$$

10.  $f(x) = x^4 - 2x^2$   $[0, 2]$

$$f(0) = 0^4 - 2(0)^2 = 0$$

$$f(2) = 2^4 - 2(2)^2 = 16 - 8 = 8$$

cannot apply Rolle's Theorem

11. Find the slope of the line between the two endpoints of the interval  $\frac{f(b)-f(a)}{b-a}$  then find the value for  $x$  that makes the  $f'(x) =$  to that slope

$$f(x) = x(x^2 - x - 2) \quad [-1, 1]$$

$$= x^3 - x^2 - 2x$$

$$\text{slope} = \frac{-2 - 0}{1 - (-1)} = \frac{-2}{2} = -1$$

$$f'(x) = 3x^2 - 2x - 2$$

$$-1 = 3x^2 - 2x - 2$$

$$0 = 3x^2 - 2x - 1$$

Quadratic Formula:

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(3)(-1)}}{2(3)}$$

$$= \frac{2 \pm \sqrt{4 + 12}}{6} = \frac{2 \pm \sqrt{16}}{6}$$

$$= \frac{2 \pm 4}{6}$$

$$x = 1, -\frac{1}{3}$$

$$12. f(x) = x^{\frac{2}{3}} = \sqrt[3]{x^2} \quad [0, 1]$$

$$\text{slope} = \frac{1-0}{1-0} = 1$$

$$f'(x) = \frac{2}{3} x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}}$$

$$3\sqrt[3]{x} \cdot 1 = \frac{2}{3\sqrt[3]{x}}$$

$$\frac{3\sqrt[3]{x}}{3} = \frac{2}{3}$$

$$\sqrt[3]{x} = \left(\frac{2}{3}\right)^3$$

$$x = \frac{8}{27}$$

13. Find the critical numbers, create intervals & choose a test value to plug into  $f'(x)$  if positive then it's increasing, if negative then decreasing.

$$f(x) = -2x^2 + 4x + 3$$

$$f'(x) = -4x + 4$$

$$0 = -4x + 4$$

$$\frac{-4}{-4} = \frac{-4x}{-4}$$

$$1 = x$$

intervals:

$$(-\infty, 1) \text{ ; } (1, \infty)$$

test values:

$$0 \text{ ; } 2$$

$$f'(0) = 4$$

$$f'(2) = -4$$

so increasing on  $(-\infty, 1)$

so decreasing on  $(1, \infty)$

$$14. \quad f(x) = (x-1)^{1/3}$$
$$f'(x) = \frac{1}{3} (x-1)^{-2/3} \cdot 1$$

$$f'(x) = \frac{1}{3 \sqrt[3]{(x-1)^2}}$$

$$3 \sqrt[3]{(x-1)^2} \cdot 0 = \frac{1}{3 \sqrt[3]{(x-1)^2}} \cdot 3 \sqrt[3]{(x-1)^2}$$

$$0 = 1$$

→  $x=1$  is undefined

$(-\infty, 1)$   $(1, \infty)$

test values: 0 & 2

$$f'(0) = \frac{1}{3 \sqrt[3]{(0-1)^2}} = \frac{1}{3 \sqrt[3]{1}} = \frac{1}{3}$$

increasing on  $(-\infty, 1)$

$$f'(2) = \frac{1}{3 \sqrt[3]{(2-1)^2}} = \frac{1}{3 \sqrt[3]{1}} = \frac{1}{3}$$

increasing on  $(1, \infty)$

15. Find all the  $x$ 's that make  $f'(x)=0$   
 Create intervals  
 Choose test values for each interval; find the sign of  $f'(x)$ .

increasing  $\rightarrow$  decreasing = maximum  
 decreasing  $\rightarrow$  increasing = minimum.

$$f(x) = x^2(3-x) = 3x^2 - x^3 \quad \text{Intervals: } (-\infty, 0) \quad (0, 2) \quad (2, \infty)$$

$$f'(x) = 6x - 3x^2$$

$$= 3x(2-x)$$

Critical values = 0, 2

Interval:	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
Test :	-1	1	3
Sign :	$f'(-1) = -9$ -	$f'(1) = 3$ +	$f'(3) = -9$ -

$x=0$  is a min, min max

$x=2$  is a max

$$16. \quad f(x) = \frac{x^3 - 3x}{3} = \frac{1}{3}x^3 - x$$

$$f'(x) = x^2 - 1 \quad \text{Intervals: } (-\infty, -1) \quad (-1, 1) \quad (1, \infty)$$

$$0 = x^2 - 1$$

$$\pm \sqrt{1} = \sqrt{x^2}$$

$$\pm 1 = x$$

Interval:	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
Test:	-2	0	2
Sign:	$f'(-2) = 3$ +	$f'(0) = -1$ -	$f'(2) = 3$ +
		max	min

$x = -1$  is a max.

$x = 1$  is a min.

17. Points of inflection are where  $f''(x) = 0$

Create intervals & test values

find the sign of  $f''(x)$  for each test value

$+$   $\rightarrow$  concave up       $-$   $\rightarrow$  concave down.

$$f(x) = 2x^3 - 3x^2 - 12x + 5$$

$$f'(x) = 6x^2 - 6x - 12$$

$$f''(x) = 12x - 6$$

$$0 = 12x - 6$$

$$\frac{1}{2} = x$$

Intervals:

$$\left(-\infty, \frac{1}{2}\right) \quad \left(\frac{1}{2}, \infty\right)$$

Interval	$\left(-\infty, \frac{1}{2}\right)$	$\left(\frac{1}{2}, \infty\right)$
test	0	1
Sign	$f''(0) = -6$	$f''(1) = 6$

concrete down on  $\left(-\infty, \frac{1}{2}\right)$   
concrete up on  $\left(\frac{1}{2}, \infty\right)$

$$18. \quad f(x) = x^3(x-4) = x^4 - 4x^3$$

$$f'(x) = 4x^3 - 12x^2$$

$$f''(x) = 12x^2 - 24x$$

$$0 = 12x(x-2)$$

$$x = 0 ; 2$$

Intervals:	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
Test :	-1	1	3
Sign of $f''(x)$	$f''(-1) = 36$ +	$f''(1) = -12$ -	$f''(3) = 36$ +

concave up on  $(-\infty, 0) \cup (2, \infty)$

concave down on  $(0, 2)$

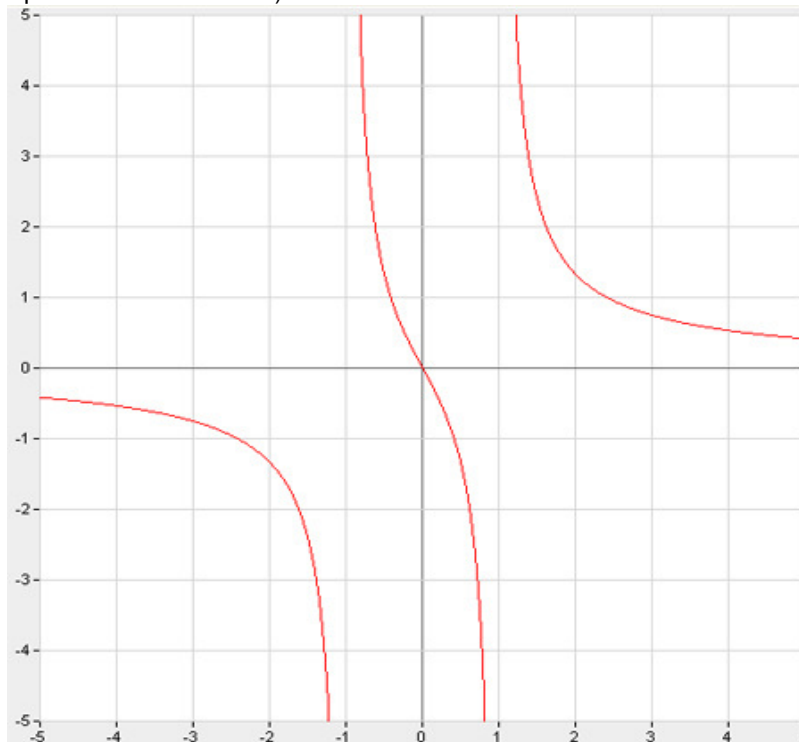
$f(x) = \frac{2x}{x^2 - 1}$	$f'(x) = \frac{-2(x^2 + 1)}{(x^2 - 1)^2}$	$f''(x) = \frac{4x(x^2 + 3)}{(x^2 - 1)^3}$	
Horizontal Asymptotes:	$y = 0$	Vertical Asymptotes:	$x = 1, -1$
X Intercepts:	$x = 0$	Y Intercepts:	$y = 0$
Critical Values:	1, -1	Points of Inflection:	0
Intervals:	$(-\infty, -1)(-1, 0)(0, 1)(1, \infty)$	Symmetry:	origin
Slant Asymptote:	none	***NOTE*** - The second derivative was incorrectly printed... SEE BELOW	

Write each interval and critical value in the Intervals column and fill out the table for each interval.

Intervals	Test Value	f(x)	Sign of f'(x)	Sign of f''(x)	Characteristic of Curve
$(-\infty, -1)$	-2	-1.333333	neg.	neg.	Decreasing, Concave Down
-1	-1	und	und	und	Vertical Asymptote
$(-1, 0)$	-0.5	1.3333	neg.	pos.**	Decreasing, Concave Up
0	0	0	neg.	0	Point of inflection
$(0, 1)$	0.5	-1.3333	neg.	neg.	Decreasing, Concave Down
1	1	und	und	und	Vertical Asymptote
$(1, \infty)$	2	1.3333	neg.	pos.	Decreasing, Concave Up

**\*\*NOTE - The second derivative was incorrectly printed as  $(x^2 - 1)^4$  it should be  $(x^2 - 1)^3$**

Graph (Show asymptotes as dotted lines):



$f(x) = x^4 - 4x^3 + 16x$	$f'(x) = (x+1)(x-2)(x-2)$	$f''(x) = 12x(x-2)$
Horizontal Asymptotes:	none	Vertical Asymptotes: none
X Intercepts:	0 and -1.68	Y Intercepts: 0
Critical Values:	-1, 2	Points of Inflection: 0 and 2
Intervals:	$(-\infty, -1)(-1, 0)(0, 2)(2, \infty)$	Symmetry: none
Slant Asymptote:	none	

Write each interval and critical value in the Intervals column and fill out the table for each interval.

Intervals	Test Value	f(x)	Sign of f'(x)	Sign of f''(x)	Characteristic of Curve
$(-\infty, -1)$	-2	16	neg.	pos.	Decreasing, Concave up
-1	-1	-11	0	pos.	Minimum
$(-1, 0)$	-0.5	-7.438	pos.	pos.	Increasing Concave Up
0	0	0	pos.	0	Point of Inflection
$(0, 2)$	1	13	pos.	neg	Increasing, Concave Down
2	2	16	0	0	Point of Inflection (Wiggle)
$(2, \infty)$	3	21	pos.	pos.	Increasing Concave Up

Graph (Show asymptotes as dotted lines):

